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Iteration for Holomorphic Maps of the Open Unit Ball and the Generalized Upper Half-Plane of C^n

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1. INTRODUCTION

Recently, Ky Fan [2, 3] extended a classical theorem of Wolff [5] on iteration of holomorphic functions in complex analysis to the theory of holomorphic functions of operators. It is our purpose here to generalize Wolff's theorem to holomorphic maps of the open unit ball B of C^n, and to that of the generalized upper half-plane E of C^n. The writing of the present paper was inspired by the results of [2, 3]. Our Theorems 1 and 2 below are analogous to the Theorems 1 and 2 in [3] in the setting of functional calculus.

A result we shall need is the following theorem of Earle and Hamilton [1]:

THEOREM (Earle–Hamilton). *If D is a non-empty bounded connected open subset of a complex Banach space, then any holomorphic map g from D strictly inside D (i.e., there exists $\varepsilon > 0$ such that $\|g(x) - y\| > \varepsilon$ for all $x \in D$, $y \notin D$) has a unique fixed point in D.*

Here by holomorphic we mean a map of D into a Banach space having a Fréchet derivative at each point of D.

2. ITERATION FOR HOLOMORPHIC SELF-MAPS OF B

In all that follows, C^n will denote the n-dimensional complex coordinate space. We take the standard Hilbert space structure on C^n. Let B be the open unit ball in C^n, i.e., $B = \{z \in C^n\colon \|z\| < 1\}$. Fix $b \in B$. Let P_b be the orthogonal projection of C^n to the subspace spanned by b, and let

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$Q_b = I - P_b$. Here I stands for the identity map of C^n. Consider the Möbius transformation

$$\varphi_b(z) = \frac{b - P_b z - (1 - \|b\|^2)^{1/2} Q_b z}{1 - \langle z, b \rangle}, \tag{1}$$

where $\langle\, , \rangle$ denotes the inner product of C^n. If $\Omega = \{z \in C^n : \langle z, b \rangle \neq 1\}$ (clearly $\Omega \supset \bar{B}$, the closure of B in C^n, since $\|b\| < 1$), then $\varphi_b : \Omega \to C^n$ is holomorphic. Furthermore, it is well known [4, Theorem 2.2.2] that φ_b is a biholomorphic map of B onto itself with $\varphi_b(0) = b$ and $\varphi_b^{-1} = \varphi_b$. In the case $\|b\| = 1$, it is easy to verify $\|\varphi_b(z)\| = 1$ for all z in B (see also identity (6) below).

The following lemma is analogous to Lemma 1 in [3], and is fundamental in the proof of Theorem 1.

LEMMA 1. *Let b be a fixed point of $\bar{B}$, $z \in B$ and $d > 0$. Then inequality*

$$\frac{|1 - \langle z, b \rangle|^2}{1 - \|z\|^2} \leqslant d \tag{2}$$

holds if and only if

$$T_b(z, d) \leqslant \frac{d(d + \|b\|^2 - 1)}{(d + \|b\|^2)^2}, \tag{3}$$

where

$$T_b(z, d) = \left\| P_b z - \frac{b}{d + \|b\|^2} \right\|^2 + \frac{d}{d + \|b\|^2} \|Q_b z\|^2. \tag{4}$$

Equality occurs in (2) *if and only if it occurs in* (3).

In the case $\|b\| < 1$, both (2) *and* (3) *are equivalent to*

$$\|\varphi_b(z)\|^2 \leqslant \frac{d + \|b\|^2 - 1}{d}; \tag{5}$$

and equalities hold simultaneously in (2), (3) *and* (5).

Proof. Since $|\langle z, b \rangle| = |\langle P_b z, b \rangle| = \|P_b z\| \ \|b\|$ and $\|z\|^2 = \|P_b z\|^2 + \|Q_b z\|^2$ for any $z \in B$, inequality (2) is equivalent to

$$\begin{aligned} 1 - \langle P_b z, b \rangle - \langle b, P_b z \rangle + \|P_b z\|^2 \|b\|^2 \\ \leqslant d(1 - \|P_b z\|^2 - \|Q_b z\|^2) \end{aligned}$$

or

$$T_b(z, d) \leqslant \frac{d-1}{d+\|b\|^2} + \frac{\|b\|^2}{(d+\|b\|^2)^2} = \frac{d(d+\|b\|^2-1)}{(d+\|b\|^2)^2}.$$

Hence (2) and (3) are equivalent to one another.

In the case $\|b\| < 1$, the equivalence of (5) and (2) (and therefore also (3)) is an immediate consequence of the identity

$$1 - \|\varphi_b(z)\|^2 = \frac{(1-\|b\|^2)(1-\|z\|^2)}{|1-\langle z, b\rangle|^2} \qquad (b \in \bar{B}, z \in B). \tag{6}$$

The above argument remains valid if each inequality sign $\leqslant$ is replaced by a strict inequality sign $<$. Consequently, equalities in (2) and (3) (and also in (5) when $\|b\| < 1$) hold simultaneously. This completes the proof.

THEOREM 1. *Let $f_1,..., f_n$ be holomorphic functions of n complex variables with domain B, and let $F = (f_1,..., f_n)$ be a holomorphic map of B into B. Let $F^{[m]}$ denote the mth iterate of F (i.e., $F^{[1]} = F$, $F^{[m]} = F \circ F^{[m-1]}$ for $m \geqslant 2$). Then there exists a w in $\bar{B}$ such that*

$$T_w(F^{[m]}(z), d(w, z)) \leqslant T_w(z, d(w, z)) = \frac{d(w, z)\{d(w, z) + \|w\|^2 - 1\}}{\{d(w, z) + \|w\|^2\}^2}, \tag{7}$$

$$\|\varphi_w(F^{[m]}(z))\| \leqslant \|\varphi_w(z)\| = \left[\frac{d(w, z) + \|w\|^2 - 1}{d(w, z)}\right]^{1/2}, \tag{8}$$

and

$$d(w, F^{[m]}(z)) \leqslant d(w, z) \tag{9}$$

hold for any $z \in B$ and $m = 1, 2,...,$ where $T_b(z, d)$ is defined by (4) *and*

$$d(w, z) = \frac{|1-\langle z, w\rangle|^2}{1-\|z\|^2}. \tag{10}$$

Furthermore, $F(w) = w$ unless $\|w\| = 1$.

Proof. Take a sequence $\{a_k\}$ such that $0 < a_k < 1$ and $\lim_{k\to\infty} a_k = 1$. Let $F_k = a_k F$. As for all $z \in B$, $y \notin B$

$$\|F_k(z) - y\| \geqslant \|y\| - a_k \|F(z)\| > 1 - a_k,$$

Earle–Hamilton's theorem implies that there exists a $z^{(k)}$ such that $F_k(z^{(k)}) = z^{(k)}$, whence $\|z^{(k)}\| < a_k < 1$. By using a subsequence if necessary, we may assume that $\{z^{(k)}\}$ tends to a limit as $k \to \infty$. Let $w = \lim_{k\to\infty} z^{(k)}$.

Then $\|w\| \leqslant 1$. Clearly, we have $F(w) = w$ unless $\|w\| = 1$. Note that the convergence $\lim_{k\to\infty} F_k(z) = F(z)$ is uniform in B, since $\|F_k(z) - F(z)\| < 1 - a_k$ for all z in B.

As $F_k(z^{(k)}) = z^{(k)}$ and $F_k(B) \subset B$, we conclude from the version of Schwarz lemma [4, p. 163] that

$$\|\varphi_{z^{(k)}}(F_k(z))\| \leqslant \|\varphi_{z^{(k)}}(z)\| \qquad (z \in B).$$

By the identity (6) with $b = z^{(k)}$, this implies

$$\frac{|1 - \langle F_k(z), z^{(k)}\rangle|^2}{1 - \|F_k(z)\|^2} \leqslant \frac{|1 - \langle z, z^{(k)}\rangle|^2}{1 - \|z\|^2} \qquad (z \in B).$$

When $k \to \infty$, we obtain the case $m = 1$ of (9) for any z in B, i.e.,

$$d(w, F(z)) \leqslant d(w, z). \tag{11}$$

Denote the mth iterate of F by $F^{[m]}$, which is also a holomorphic map of B into B. Since $F^{[m]}(z) = F(F^{[m-1]}(z))$ and (11) is proved for any z in B, we have

$$d(w, F^{[m]}(z)) \leqslant d(w, F^{[m-1]}(z)) \leqslant \cdots \leqslant d(w, z)$$

for any z in B and $m = 1, 2, \ldots$, so that we obtain (9). Then, by an application of Lemma 1, (7) and the case of $\|w\| < 1$ of (8) follow from (9) and (10). When $\|w\| = 1$, (8) remains true and the inequality therein becomes equality according to the identity (6). This completes the proof of Theorem 1.

Remark 1. Using (10) and the identity

$$\begin{aligned} &1 - 2\operatorname{Re}\langle z, w\rangle + \|w\|^2(1 - \|Q_w z\|^2 \\ &\quad = |1 - \langle z, w\rangle|^2 + \|w\|^2(1 - \|z\|^2), \end{aligned} \tag{12}$$

we can give another expression for the quantity on the right side of (7)

$$\begin{aligned} &\frac{d(w,z)(d(w,z) + \|w\|^2 - 1)}{(d(w,z) + \|w\|^2)^2} \\ &\quad = \|\varphi_w(z)\|^2 \frac{|1 - \langle z, w\rangle|^4}{\{1 - 2\operatorname{Re}\langle z, w\rangle + \|w\|^2(1 - \|Q_w z\|^2\}^2}. \end{aligned} \tag{13}$$

In Theorem 1, if F is fixed-point-free in B, then we must have $\|w\| = 1$, and (7) (combined with (4), (10), (12) and (13)) gives

$$\begin{aligned} &\|P_w(F^{[m]}(z)) - c(w,z)\|^2 + r(w,z)\|Q_w(F^{[m]}(z))\|^2 \\ &\quad \leqslant \|P_w z - c(w,z)\|^2 + r(w,z)\|Q_w z\|^2 = r^2(w,z), \end{aligned}$$

for any z in B and $m = 1, 2, \ldots$, where

$$c(w, z) = \frac{(1 - \|z\|^2)w}{|1 - \langle z, w\rangle|^2 + 1 - \|z\|^2},$$

$$r(w, z) = \frac{|1 - \langle z, w\rangle|^2}{|1 - \langle z, w\rangle|^2 + 1 - \|z\|^2}.$$

Then, $\|c(w, z)\| + r(w, z) = 1$ for all z in B and all w in C^n with $\|w\| = 1$. In the simplest case when $n = 1$, it is seen that $P_w = I$ and $Q_w = 0$. If F is fixed-point-free in the open unit disk of the complex plane, by identity (12) we have

$$c(w, z) = \frac{(1 - \|z\|^2)w}{2 - 2\,\mathrm{Re}(\bar{w}z)},$$

$$r(w, z) = \frac{|1 - \bar{w}z|^2}{2 - \mathrm{Re}(\bar{w}z)}.$$

Hence in this case Theorem 1 reduces to the classical Wolff's theorem.

Remark 2. In the preceding theorem, if $\|w\| = 1$, then

$$\lim_{s \uparrow 1} \langle F^{[m]}(sw), w\rangle = 1 \qquad (m = 1, 2, \ldots). \tag{14}$$

Indeed, when $\|w\| = 1$, (9) with $z = sw$ $(0 < s < 1)$ and (10) give

$$|1 - \langle F^{[m]}(sw), w\rangle|^2 \leqslant \frac{(1 - s)^2}{1 - s^2}(1 - \|F^{[m]}(sw)\|^2) \leqslant \frac{1 - s}{1 + s}.$$

In the case of $n = 1$, since $\langle F(sw), w\rangle = w^{-1}F(sw)$, (14) with $m = 1$ becomes

$$\lim_{s \uparrow 1} F(sw) = w,$$

which is the same as a result pointed out by T. Ando (see [3, Remark 3]).

3. Iteration for Holomorphic Self-Maps of E

Let E denote the generalized upper half-plane of C^n, i.e., the set of all $z \in C^n$ with $\mathrm{IM}\,z > 0$, where $\mathrm{IM}\,z = \mathrm{Im}\,z_1 - \|z'\|^2$, $z = (z_1, z')$, $z' = (z_2, \ldots, z_n)$ and $\|z'\|^2 = |z_2|^2 + \cdots + |z_n|^2$. Then, E is an unbounded convex domain in C^n, and $\bar{E} = \{z \in C^n : \mathrm{IM}\,z \geqslant 0\}$. Consider the Cayley transformation Φ defined by

$$u = \Phi(z) = i\,\frac{e_1 + z}{1 - z_1}, \tag{15}$$

where $e_1 = (1, 0')$. This sends z in C^n with $z_1 \neq 1$ to u in C^n, and implies

$$\mathrm{IM}\, u = \frac{1 - \|z\|^2}{|1 - z_1|^2}, \tag{16}$$

and

$$z = \Psi(u) = \frac{2u}{i + u_1} - e_1. \tag{17}$$

Hence Φ is a biholomorphic map of B onto E (see [4, p. 31]).

In this section, we shall consider iterations of holomorphic self-maps of E. To prove Theorem 2 below we shall need the following

LEMMA 2. *Let $v = (v_1, v')$ be a fixed vector of $\bar{E}$, $u = (u_1, u') \in E$ and $h > 0$. Then inequality*

$$\frac{|i(u_1 - \bar{v}_1) + 2\langle u', v'\rangle|^2}{\mathrm{IM}\, u} \leqslant h \tag{18}$$

holds if and only if

$$R_v(u, h, h + 4\|v'\|^2) \leqslant \frac{h - 4\,\mathrm{IM}\, v}{4(h + \|v'\|^2)}, \tag{19}$$

where

$$R_v(u, h, t) = h^{-1}\|P_{v'}u' + 2it^{-1}(u_1 - \bar{v}_1)v'\|^2 + t^{-1}\|Q_{v'}u'\|^2 + t^{-2}|(u_1 - \bar{v}_1) - \tfrac{1}{2}it|^2 \tag{20}$$

for $u \in E$ and $t > 0$. Equality occurs in (18) *if and only if it occurs in* (19).

Proof. Note that $\|u'\|^2 = \|P_{v'}u'\|^2 + \|Q_{v'}u'\|^2$ and $|\langle u', v'\rangle| = \|P_{v'}u'\|\,\|v'\|$. By multiplying out everything we see that (18) is equivalent to

$$(h + 4\|v'\|^2)\|P_{v'}u'\|^2 + 2i(u_1 - \bar{v}_1)\langle v', P_{v'}u'\rangle - 2i(\bar{u}_1 - v_1)\langle P_{v'}u', v'\rangle$$
$$+ h\|Q_{v'}u'\|^2 + |u_1|^2 - u_1(v_1 - \tfrac{1}{2}ih) - \bar{u}_1(\bar{v}_1 + \tfrac{1}{2}ih) + |v_1|^2 \leqslant 0.$$

By completing the square, the last inequality can be rewritten as (19).

The above argument remains true if every inequality sign $\leqslant$ is replaced by a strict inequality sign $<$. Hence, equalities in (18) and (19) hold simultaneously. This proves the lemma.

THEOREM 2. *Let $g_1, \ldots, g_n$ be holomorphic functions of n complex* *variables with domain E. Let* $G = (g_1, \ldots, g_n)$ *be a holomorphic map of E into itself. Then either*

$$\mathrm{IM}\, G(u) \geqslant \mathrm{IM}\, u \tag{21}$$

holds for any $u \in E$; *or there exists a* $v = (v_1, v')$ *in* $\bar{E}$ *such that*

$$\begin{aligned} R_v(G^{[m]}(u), h, h + 4\,\|v'\|^2) &\leqslant R_v(u, h, h + 4\,\|v'\|^2) \\ &= \frac{h - 4\mathrm{IM}\, v}{4(h + \|v'\|^2)} \end{aligned} \tag{22}$$

and

$$h(v, G^{[m]}(u)) \leqslant h(v, u) \tag{23}$$

hold for any u in E and $m = 1, 2, \ldots,$ *where* $R_v(u, h, t)$ *is defined by* (20) *and*

$$h = h(v, u) = \frac{|i(u_1 - \bar{v}_1) + 2\langle u', v'\rangle|^2}{\mathrm{IM}\, u}. \tag{24}$$

Furthermore, $G(v) = v$ *unless* $\mathrm{IM}\, v = 0$.

Proof. Consider the Cayley transformation Φ and its holomorphic inverse Ψ defined by (15) and (17), respectively. Then $\Phi(B) = E$ and $\Psi(E) = B$. It is seen that $F = \Psi \circ G \circ \Phi$ is a holomorphic map such that $F(B) \subset B$, so by Theorem 1 there exists a w in $\bar{B}$ having the properties stated in Theorem 1. Put $\Phi(w) = v$. Since $F(z) = \Psi(G(u))$ for $z \in B$, we have by (17)

$$F(z) = \frac{2G(u)}{i + g_1(u)} - e_1. \tag{25}$$

Thus

$$1 - \langle F(z), e_1\rangle = \frac{2i}{i + g_1(u)} \tag{26}$$

and, in particular,

$$1 - z_1 = \frac{2i}{i + u_1}. \tag{27}$$

It is convenient to distinguish two cases.

Case 1. $w = e_1$. By (9) with $m = 1$, we then obtain

$$\frac{|1 - \langle F(z), e_1\rangle|^2}{1 - \|F(z)\|^2} \leqslant \frac{|1 - z_1|^2}{1 - \|z\|^2} \qquad (z \in B).$$

From (16), (26) and (27) it follows that the last inequality can be rewritten as

$$(\mathrm{IM}\, G(u))^{-1} \leqslant (\mathrm{IM}\, u)^{-1} \qquad (u \in E),$$

which is equivalent to (21).

Case 2. $w \neq e_1$. Then, by (15) and (16), $v = \Phi(w)$ is finite, and $\mathrm{IM}\, v > 0$ if and only if $\|w\| < 1$; $\mathrm{IM}\, v = 0$ if $\|w\| = 1$ but $w \neq e_1$. Since $F(w) = w$ unless $\|w\| = 1$ by Theorem 1, we have $G(v) = v$ unless $\mathrm{IM}\, v = 0$. Simple computations show that for $z \in B$

$$|1 - \langle F(z), w\rangle|^2 = \frac{4\,|i(g_1(u) - \bar{v}_1) + 2\langle G(u)', v'\rangle|^2}{|i + g_1(u)|^2\,|i + v_1|^2} > 0 \tag{28}$$

and

$$|1 - \langle z, w\rangle|^2 = \frac{4\,|i(u_1 - \bar{v}_1) + 2\langle u', v'\rangle|^2}{|i + u_1|^2\,|i + v_1|^2} > 0, \tag{29}$$

where $G(u)' = (g_2(u), \ldots, g_n(u))$. Thus by (16), (26), (27), (28) and (29), (9) with $m = 1$ becomes

$$h(v, G(u)) \leqslant h(v, u) \qquad (u \in E), \tag{30}$$

where $h(v, u)$ is defined by (24). It follows from Lemma 2 that the case of $m = 1$ of (22) holds for any $u \in E$. Suppose as before that $G^{[m]}$ is the mth iterate of G. Obviously, $G^{[m]}(E) \subset E$. Since (30) is proved for any $u \in E$, and $G^{[m]}(u) = G(G^{[m-1]}(u))$, we have

$$h(v, G^{[m]}(u)) \leqslant h(v, G^{[m-1]}(u)) \leqslant \cdots \leqslant h(v, u) \qquad (u \in E).$$

Then, (22) follows easily from Lemma 2. Theorem 2 is proved.

Remark 3. In the simplest case when $n = 1$, (21), (22) and (23) in the preceding theorem become respectively

$$\mathrm{Im}\, g_1(u) \geqslant \mathrm{Im}\, u_1,$$

$$|g_1^{[m]}(u) - (\bar{v}_1 + \tfrac{1}{2} i h_1)|^2 \leqslant |u_1 - (\bar{v}_1 + \tfrac{1}{2} i h_1)|^2 = \tfrac{1}{4} h_1^2 - h_1\, \mathrm{Im}\, v_1,$$

and

$$h_1(v_1, g_1^{[m]}(u)) \leqslant h_1(v_1, u_1).$$

Here $h_1 = h_1(v_1, u_1) = |u_1 - \bar{v}_1|^2/\mathrm{Im}\, u_1$.

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